Flats in spaces with convex geodesic bicombings

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Abstract

In spaces of nonpositive curvature the existence of isometrically embedded flat (hyper)planes is often granted by apparently weaker conditions on large scales. We show that some such results remain valid for metric spaces with non-unique geodesic segments under suitable convexity assumptions on the distance function along distinguished geodesics. The discussion includes, among other things, the Flat Torus Theorem and Gromov's hyperbolicity criterion referring to embedded planes. This generalizes results of Bowditch for Busemann spaces.

1 Introduction

The geometry of spaces of global nonpositive curvature is largely dominated by the convexity of the distance function. Thus a considerable part of the theory of CAT(0) spaces [2, 7] carries over to Busemann spaces [8, 27] (defined by the property that $d\circ(\sigma_1, \sigma_2)$ is convex for any pair of constant speed geodesics σ_1, σ_2 parametrized on the same interval). However, this larger class of spaces has the defect of not being preserved under limit processes. For example, among normed real vector spaces, exactly those with strictly convex norm satisfy the Busemann property, and a sequence of such norms on $\mathbb{R}^n$, say, may converge to a non-strictly convex norm. This motivates the study of an even weaker notion of nonpositive curvature that dispenses with the uniqueness of geodesics but retains the convexity condition for a suitable selection of geodesics (compare Sect. 10 in [23]). In any normed space, the affine segments $t \mapsto (1-t)x + ty$ ($t \in [0,1]$) provide such a choice. In particular, the relaxed condition carries the potential for simultaneous generalizations of results for nonpositively curved and Banach spaces. Another reason for this investigation is that l_1- and l_∞-type metrics have been put in use in geometric group theory; see, for example, [3, 5, 9, 24]. The recent paper [22] shows that symmetric spaces of noncompact type possess natural, non-strictly convex Finsler metrics adapted to the geometry of Weyl chambers and pertinent to the dynamics at infinity.

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In a previous article we initiated a systematic study of spaces of weak global nonpositive curvature as described above, with the main objective of providing geometric models of this type for hyperbolic groups; see [11] and in particular Theorem 1.3 therein. The purpose of the present paper is to carry the analogy with CAT(0) and Busemann spaces further with regard to existence results for flat subspaces. Here, for a metric space $X = (X, d)$, a map $\sigma \colon X \times X \times [0,1] \to X$ will be simply called a *bicombing* if the respective family of maps $\sigma_{xy} := \sigma(x, y, \cdot) \colon [0,1] \to X$ satisfies the following three properties:

(i) σ_{xy} is a geodesic from x to y, that is, $\sigma_{xy}(0) = x$, $\sigma_{xy}(1) = y$, and $d(\sigma_{xy}(t), \sigma_{xy}(t')) = |t - t'|\, d(x, y)$ for $t, t' \in [0,1]$;

(ii) $\sigma_{yx}(t) = \sigma_{xy}(1 - t)$ for $t \in [0,1]$;

(iii) $d(\sigma_{xy}(t), \sigma_{x'y'}(t)) \le (1 - t)\, d(x, x') + t\, d(y, y')$ for $t \in [0,1]$.

(This corresponds to a conical and reversible geodesic bicombing in the terminology of [11].) Notice that these conditions do not ensure that $t \mapsto d(\sigma_{xy}(t), \sigma_{x'y'}(t))$ is a *convex* function on $[0,1]$. This is guaranteed under the following extra assumption on the traces:

(iv) $\mathrm{im}(\sigma_{pq}) \subset \mathrm{im}(\sigma_{xy})$ whenever $p = \sigma_{xy}(r)$ and $q = \sigma_{xy}(s)$ with $r \le s$.

Note that then $\sigma_{pq}(t) = \sigma_{xy}((1-t)r + ts)$ for $t \in [0,1]$ by (i). A bicombing σ satisfying (iv) will be called *consistent*. Busemann spaces and convex subsets of normed spaces possess consistent bicombings, whereas some additional examples of (general) bicombings are obtained via 1-Lipschitz retractions onto subspaces. We refer to [11] for more information.

Our first main result is the following generalization of the hyperbolicity criterion for cocompact CAT(0) spaces stated on p. 119 in [20]. A detailed proof of Gromov's result, inspired by [13], was given in [6]. For the case of Busemann spaces, both Theorem 1.1 and Theorem 1.2 below were shown by Bowditch [4].

Theorem 1.1 (Flat plane). *Let X be a proper metric space with a consistent bicombing σ and with cocompact isometry group. Then X is hyperbolic if and only if X does not contain an isometrically embedded normed plane.*

Another well-known result from the theory of spaces of nonpositive curvature is the Flat Torus Theorem, originally proved for smooth manifolds in [19, 25] (see also [28, 14] for some earlier contributions in this direction). A detailed account of this result and its applications in the context of CAT(0) spaces is given in Chap. II.7 of [7]. We have:

Theorem 1.2 (Flat torus). *Let X be a proper metric space with a consistent bicombing σ. Let Γ be a group acting properly and cocompactly by isometries on X, and suppose that σ is Γ-equivariant. If Γ has a free abelian*

subgroup group A of rank $n \geq 1$, then X contains an isometrically embedded n-dimensional normed space on which A acts by translations.

Here, σ being Γ-*equivariant* means that $\gamma \circ \sigma_{xy} = \sigma_{\gamma(x)\gamma(y)}$ for all $\gamma \in \Gamma$ and $(x, y) \in X \times X$. For example, beyond uniquely geodesic spaces, every injective metric space (or absolute 1-Lipschitz retract) X admits a bicombing σ that is equivariant with respect to the full isometry group $\mathrm{Isom}(X)$ of X; see Proposition 3.8 in [24]. Furthermore, it is shown in [11] that every proper metric space X with a bicombing and with finite combinatorial dimension in the sense of [12] also admits a *unique consistent* bicombing, which is $\mathrm{Isom}(X)$-equivariant, too.

We briefly introduce some terminology that will be used throughout the paper. Let X be a metric space. A map $\xi \colon I \to X$ of some interval $I \subset \mathbb{R}$ is a *geodesic* if there is a constant $c \geq 0$, the *speed* of ξ, such that $d(\xi(t), \xi(t')) = c|t - t'|$ for all $t, t' \in I$. A *line* or a *ray* in X is a unit speed geodesic defined on $\mathbb{R}$ or $\mathbb{R}_+ := [0, \infty)$, respectively. Two lines ξ, ξ' are *parallel* if $\sup_{s \in \mathbb{R}} d(\xi(s), \xi'(s)) < \infty$, and two rays η, η' are *asymptotic* if $\sup_{s \in \mathbb{R}_+} d(\eta(s), \eta'(s)) < \infty$. A family of geodesics $\xi_a \colon I_a \to X$ indexed by a set A will be called *coherent* if $t \mapsto d(\xi_a(\alpha(t)), \xi_b(\beta(t)))$ is a convex function on $[0, 1]$ whenever $a, b \in A$ and $\alpha \colon [0, 1] \to I_a$ and $\beta \colon [0, 1] \to I_b$ are affine maps[1]. Notice that if σ is a consistent bicombing on X and $A \subset X \times X$ is any set, then $\{\sigma_{xy} : (x, y) \in A\}$ is a coherent family. Given a bicombing σ on X, we shall often write $[x, y](t)$ for $\sigma_{xy}(t)$ and $[x, y]$ for $\mathrm{im}(\sigma_{xy})$ without further comment. A set $C \subset X$ will be called σ-*convex* if $[x, y] \subset C$ whenever $x, y \in C$. The *(closed)* σ-*convex hull* of a subset $S \subset X$ is the smallest (closed) σ-convex set containing S. A line $\xi \colon \mathbb{R} \to X$ will be called a σ-*line* if its trace is σ-convex; equivalently, $[\xi(r), \xi(s)](t) = \xi((1 - t)r + ts)$ for all $r, s \in \mathbb{R}$ and $t \in [0, 1]$. For two σ-lines ξ, ξ' the function $s \mapsto d(\xi(s), \xi'(s))$ is convex, hence constant in case ξ, ξ' are parallel.

The paper is organized as follows. In Sect. 2 we discuss a generalized Flat Strip Theorem. Unlike for Busemann spaces, the σ-convex hull of a pair of parallel σ-lines may be "thick" and the two lines may span different, though pairwise isometric, flat (normed) strips. We also give a criterion for the existence of an embedded normed half-plane. This is then used in Sect. 3 for the proof of Theorem 1.1. A variant of this result for injective metric spaces is also shown. In Sect. 4 we establish basic properties of semi-simple isometries of spaces with bicombings. We employ a barycenter map for finite subsets which was introduced in the context of Busemann spaces in [15]. Sect. 5 addresses the question whether a hyperbolic (axial) isometry of a metric space with a consistent bicombing σ also possesses an axis that is at the same time a σ-line. This is false in general but holds true for the hyperbolic elements of a group Γ as in Theorem 1.2. As an auxiliary tool we use a fixed point theorem for nonexpansive mappings proved originally for

[1] α, β are neither assumed to be surjective nor orientation preserving.

Banach spaces in [18]. Finally, Sect. 6 contains the proof of Theorem 1.2, and we conclude by an example in which the embedded flat cannot be chosen to be σ-convex.

In a subsequent paper [10] by the first author it is shown that a proper cocompact metric space X with a (possibly non-consistent) bicombing contains an isometric copy of the n-dimensional normed space V under the following asymptotic condition, also studied in [29]: there exist subsets $S_k \subset X$ and a sequence $0 < R_k \to \infty$ such that the rescaled sets $(S_k, R_k^{-1}d)$ converge in the Gromov–Hausdorff topology to the unit ball of V. This generalizes a result of Kleiner for Busemann spaces; see Proposition 10.22 and the more comprehensive Theorem D in [23]. Likewise, it follows that a proper cocompact metric space X with a bicombing contains a flat (normed) n-plane whenever there is a quasi-isometric embedding of $\mathbb{R}^n$ into X, a result which was first shown for manifolds of nonpositive curvature in [1]. Using these more recent results from [10] one can extend Theorem 1.1 and Theorem 1.2, except possibly for the fact that the subgroup A acts on the embedded flat, to the case of general bicombings. It is not clear, however, whether this yields significant improvements. In fact, it is still an open problem whether there exists a metric space, proper or not, that admits a bicombing but no consistent bicombing. In any case, the arguments presented here are more direct. If the (consistent) bicombing σ in Theorem 1.1 is equivariant with respect to a cocompact group of isometries of X, and X is non-hyperbolic, then the construction we describe produces an embedded normed plane that is foliated by mutually parallel σ-lines in at least one direction.

2 Flat strips and half-planes

We start with the construction of an embedded flat strip in an arbitrary metric space, using only a minimal coherent family of geodesics, as defined in the introduction.

Proposition 2.1 (Flat strip)**.** *Let X be a metric space. Suppose that $\{\xi, \xi'\} \cup \{\eta_s : s \in \mathbb{R}\}$ is a coherent collection of geodesics in X, where ξ, ξ' are two parallel lines with disjoint images and $\eta_s \colon [0,1] \to X$ is a geodesic from $\xi(s)$ to $\xi'(s)$ for every $s \in \mathbb{R}$. Then the map*

$$f \colon \mathbb{R} \times [0,1] \to X, \quad f(s,t) = \eta_s(t),$$

is an isometric embedding with respect to the metric on $\mathbb{R} \times [0,1]$ induced by some norm on $\mathbb{R}^2$.

Proof. For $r \in \mathbb{R}$, put $\nu(r) := d(\xi(0), \xi'(r))$. We have $d(\xi(R), \xi'(R + r)) = \nu(r)$ for every $R \in \mathbb{R}$ since the left hand side is a convex non-negative bounded function of R, hence constant. We claim that for every pair of

points $p = (s, t)$ and $p' = (s + \Delta s, t + \Delta t)$ in $\mathbb{R} \times [0, 1]$ we have

$$d(f(p), f(p')) = \begin{cases} |\Delta t|\, \nu\bigl(\frac{\Delta s}{\Delta t}\bigr) & \text{if } \Delta t \neq 0, \\ |\Delta s| & \text{if } \Delta t = 0. \end{cases} \tag{2.1}$$

There is no loss of generality in assuming $\Delta t \geq 0$. Suppose first that $\Delta t > 0$, and put $r := \frac{\Delta s}{\Delta t}$. Let $q := (s - tr, 0)$ and $q' := (s + (1 - t)r, 1)$ denote the points where the line through p, p' intersects $\mathbb{R} \times \{0\}$ and $\mathbb{R} \times \{1\}$. Then

$$d(f(q), f(q')) = d(\xi(s - tr), \xi'(s - tr + r)) = \nu(r). \tag{2.2}$$

We put $\eta := \eta_s$ and $\eta' := \eta_{s+\Delta s}$. By convexity, we get that

$$d(f(q), f(p)) = d(\xi(s - tr), \eta(t)) \leq t\, d(\xi(s - r), \eta(1)) = t\, \nu(r). \tag{2.3}$$

Likewise, we have

$$d(f(p'), f(q')) \leq (1 - t - \Delta t)\, \nu(r) \tag{2.4}$$

as well as $d(\eta(0), \eta'(\Delta t)) \leq \Delta t\, \nu(r)$ and $d(\eta(1-\Delta t), \eta'(1)) \leq \Delta t\, \nu(r)$. Hence, by the convexity of $\lambda \mapsto d(\eta(\lambda), \eta'(\lambda + \Delta t))$ on $[0, 1 - \Delta t]$, also

$$d(f(p), f(p')) = d(\eta(t), \eta'(t + \Delta t)) \leq \Delta t\, \nu(r). \tag{2.5}$$

From (2.2)–(2.5) and the triangle inequality we see that all inequalities derived so far are in fact equalities. In view of (2.5), this shows in particular the first part of (2.1). The second case follows by continuity from the first, since $\bigl||r| - \nu(0)\bigr| \leq \nu(r) \leq \nu(0) + |r|$ for all $r \in \mathbb{R}$ and hence

$$\lim_{\Delta t \to 0} |\Delta t|\, \nu\bigl(\tfrac{\Delta s}{\Delta t}\bigr) = |\Delta s|.$$

Now, to conclude the proof, note that $\nu(r) > 0$ for all $r \in \mathbb{R}$, as ξ and ξ' have disjoint images. It then follows readily from (2.1) that there is a norm $\|\cdot\|$ on $\mathbb{R}^2$ such that $d(f(p), f(p')) = \|p' - p\|$ for all $p, p' \in \mathbb{R} \times [0, 1]$. Note that the triangle inequality for $\|\cdot\|$ is just inherited from X. $\square$

The following example shows that, in general, if we replace ξ by $s \mapsto \xi(s + a)$ for some $a \neq 0$, we may get a different strip in X.

Example 2.2. Define piecewise affine functions $g, \bar{g} \colon \mathbb{R} \times [0, 1] \to \mathbb{R}$ such that

$$g(s, t) = \begin{cases} \frac{1}{2}t & \text{if } s \leq 0, \\ \frac{1}{2}|s - t| & \text{if } 0 \leq s \leq 1, \\ \frac{1}{2}(1 - t) & \text{if } s \geq 1, \end{cases}$$

and $\bar{g}(s,t) = \frac{1}{2} - g(s, 1-t)$. Note that $g = \bar{g}$ outside of $(0,1)^2$, whereas the graphs of g and $\bar{g}$ over $[0,1]^2$ bound a simplex in $\mathbb{R}^3$. Consider the sets $Y = \mathbb{R} \times [0,1] \times \mathbb{R}$ and

$$X := \{(s,t,u) \in Y : g(s,t) \le u \le \bar{g}(s,t)\},$$

both equipped with the metric induced by the maximum norm on $\mathbb{R}^3$. Let $\bar{\sigma}\colon (x,y,\lambda) \mapsto (1-\lambda)x + \lambda y$ be the canonical bicombing on Y. The vertical retraction $\pi\colon (s,t,u) \mapsto (s,t,\min\{\max\{u, g(s,t)\}, \bar{g}(s,t)\})$ from Y onto X is 1-Lipschitz. It follows from results in [11] (see Lemma 2.1, Theorem 1.1, the observation at the end of Sect. 2, and Theorem 1.2) that $\tilde{\sigma} := \pi \circ \bar{\sigma}|_{X \times X \times [0,1]}$ is a bicombing on X and that X possesses a unique consistent bicombing σ, which furthermore satisfies $\sigma_{xy} = \tilde{\sigma}_{xy}$ whenever the consistency condition (iv) stated in the introduction holds for $\tilde{\sigma}_{xy}$. In particular, the geodesics $\xi, \xi'\colon \mathbb{R} \to X$ defined by $\xi(s) := (s, 0, g(s,0))$ and $\xi'(s) := (s, 1, g(s,1))$ are two (parallel) σ-lines. It is then not difficult to see that the strip formed by the segments $\sigma_{\xi(s)\xi'(s+1)}$ corresponds to the graph of g, whereas the segments $\sigma_{\xi(s+1)\xi'(s)}$ trace out the graph of $\bar{g}$.

We also see that in Proposition 2.1, for fixed $t \in (0,1)$, the lines $s \mapsto f(s,t)$ need not be σ-lines in general: clearly the lines $s \mapsto \sigma_{\xi(s)\xi'(s+1)}(\frac{1}{2})$ and $s \mapsto \sigma_{\xi(s+1)\xi'(s)}(\frac{1}{2})$ in the above example cannot both be σ-lines. However, it is not difficult to deduce the following result.

Theorem 2.3 (Flat strips). *Let X be a metric space with a consistent bicombing σ, and let $\xi, \xi'\colon \mathbb{R} \to X$ be two parallel σ-lines with disjoint traces. Then there exists a norm on $\mathbb{R}^2$ such that the following assertions hold for the metric it induces on $\mathbb{R} \times [0,1]$:*

(1) *For every $a \in \mathbb{R}$, the map $f_a\colon \mathbb{R} \times [0,1] \to X$ satisfying $f_a(s + ta, t) = \sigma_{\xi(s)\xi'(s+a)}(t)$ for all $(s,t) \in \mathbb{R} \times [0,1]$ is an isometric embedding.*

(2) *If, in addition, X is proper, there also exists an isometric embedding $f\colon \mathbb{R} \times [0,1] \to X$ such that $f(\cdot, 0) = \xi$, $f(\cdot, 1) = \xi'$, and $s \mapsto f(s,t)$ is a σ-line parallel to ξ and ξ' for every fixed $t \in (0,1)$.*

For a corresponding (but easier) result in the case of Busemann spaces, see Lemma 1.1 and the remark thereafter in [4] (compare also Proposition 5.3 in [16]). Part (1) is a direct consequence of Proposition 2.1, and (2) then follows by a simple limit argument (notice that in (1), all f_a satisfy $f_a(\cdot, 0) = \xi$ and $f_a(\cdot, 1) = \xi'$). As this result will not be used in the sequel, we omit the details.

We now proceed to an existence result for embedded flat half-planes, which will be instrumental in the proof of Theorem 1.1. We need the following analogue of the Tits cone in the case of CAT(0) spaces. Let $\mathscr{R}$ be a coherent collection of rays in X, and denote by $\mathscr{R}(\infty)$ the set of equivalence

classes of mutually asymptotic rays in $\mathscr{R}$. For $(a,\xi),(b,\eta) \in \mathbb{R}_+ \times \mathscr{R}$, we put

$$d_\infty((a,\xi),(b,\eta)) := \lim_{\lambda\to\infty} \frac{1}{\lambda}\, d(\xi(a\lambda),\eta(b\lambda)).$$

Note that the limit exists by convexity, and $|a-b| \le d_\infty((a,\xi),(b,\eta)) \le a+b$. This defines a pseudometric d_∞ on $\mathbb{R}_+ \times \mathscr{R}$, and the respective quotient metric space is a metric cone over $\mathscr{R}(\infty)$ (compare [2], p. 38). In particular, for $a > 0$, $d_\infty((a,\xi),(a,\eta)) = a\, d_\infty((1,\xi),(1,\eta))$, and this is zero if and only ξ and η are asymptotic. The following result should now be compared with Proposition II.4.2 in [2] and Proposition II.9.8 and Corollary II.9.9 in [7].

Proposition 2.4 (Flat half-plane)**.** *Let X be a metric space. Suppose that $\{\xi\} \cup \{\eta_s : s \in \mathbb{R}\}$ is a coherent collection of geodesics in X, where ξ is a line and every η_s is a ray with $\eta_s(0) = \xi(s)$ asymptotic to $\eta := \eta_0$. Then, for all $a,b > 0$, the function $s \mapsto d(\xi(s+a),\eta_s(b))$ is non-decreasing on $\mathbb{R}$ with limit*

$$\lim_{s\to\infty} d(\xi(s+a),\eta_s(b)) = d_\infty((a,\xi),(b,\eta)). \tag{2.6}$$

Furthermore, if for every $a \in \mathbb{R}$ the function $s \mapsto d(\xi(s+a),\eta_s(1))$ is constant on $\mathbb{R}$ and nonzero, then the map

$$f\colon \mathbb{R}\times\mathbb{R}_+ \to X, \quad f(s,t) := \eta_s(t),$$

is an isometric embedding with respect to the metric on $\mathbb{R}\times\mathbb{R}_+$ induced by some norm on $\mathbb{R}^2$.

Proof. Let $a,b > 0$. First we show that for all $0 < r \le \lambda \le r+1$,

$$d(\xi(ar+a),\eta_{ar}(b)) \ge \frac{1}{\lambda}\, d(\xi(a\lambda),\eta(b\lambda)). \tag{2.7}$$

Since η_{ar} and the ray $t \mapsto \eta(br+t)$ are asymptotic, we have

$$d(\eta_{ar}(b),\eta(br+b)) \le d(\eta_{ar}(0),\eta(br)) = d(\xi(ar),\eta(br)). \tag{2.8}$$

It follows that

$$\begin{aligned} d(\xi(ar+a),\eta_{ar}(b)) &\ge d(\xi(ar+a),\eta(br+b)) - d(\xi(ar),\eta(br)) \\ &\ge \Big(\frac{r+1}{\lambda} - \frac{r}{\lambda}\Big) d(\xi(a\lambda),\eta(b\lambda)), \end{aligned}$$

which is (2.7). Putting $\lambda = 1$ we get $d(\xi(ra+a),\eta_{ra}(b)) \ge d(\xi(a),\eta_0(b))$. Likewise, for all $s \in \mathbb{R}$ and $0 < r \le 1$,

$$d(\xi(s+ra+a),\eta_{s+ra}(b)) \ge d(\xi(s+a),\eta_s(b)),$$

so $s \mapsto d(\xi(s+a),\eta_s(b))$ is non-decreasing. Furthermore, for all $s \in \mathbb{R}$ and $\lambda \ge 1$, we have

$$d(\xi(s+a),\eta_s(b)) \le \frac{1}{\lambda}\, d(\xi(s+a\lambda),\eta_s(b\lambda)).$$

Together with (2.7), this gives (2.6).

For the second part of the proposition we have that $s \mapsto d(\xi(s+a), \eta_s(1))$ is constant for every $a \in \mathbb{R}$ and that these values $\nu(a) := d(\xi(a), \eta_0(1))$ are all positive. We first claim that

$$d(\xi(s+ta), \eta_s(t)) = t\,\nu(a) \tag{2.9}$$

for all $t \geq 0$. The left hand side is convex as a function of t, thus it suffices to show this equality for $0 \leq t \in \mathbb{Z}$. For $t = 0, 1$, (2.9) clearly holds. Consequently, by convexity, $d(\xi(s+ta), \eta_s(t)) \geq t\,\nu(a)$ for all $t > 1$. The reverse inequality for $1 < t \in \mathbb{Z}$ follows by the triangle inequality since

$$d(\eta_{s+ka}(t-k), \eta_{s+ka-a}(t-k+1)) \leq d(\eta_{s+ka}(0), \eta_{s+ka-a}(1)) = \nu(a)$$

for $k = t, t-1, \dots, 1$ (compare (2.8)). Next, we claim that for every pair of points $p = (s,t)$ and $p' = (s+\Delta s, t+\Delta t)$ in $\mathbb{R} \times \mathbb{R}_+$ we have

$$d(f(p), f(p')) = \begin{cases} |\Delta t|\,\nu\big(-\frac{\Delta s}{\Delta t}\big) & \text{if } \Delta t \neq 0, \\ |\Delta s| & \text{if } \Delta t = 0, \end{cases}$$

similarly as in the proof of Proposition 2.1. To show this, suppose without loss of generality that $\Delta t \geq 0$. Let first $\Delta t > 0$, and put $a := \frac{\Delta s}{\Delta t}$ and $q := (s - ta, 0)$. Then (2.9) yields

$$\begin{aligned} d(f(p), f(p')) &= d(\eta_s(t), \eta_{s+\Delta s}(t+\Delta t)) \\ &\leq d(\eta_s(0), \eta_{s+\Delta s}(\Delta t)) \\ &= \Delta t\,\nu(-a) \end{aligned}$$

as well as $d(f(q), f(p)) = t\,\nu(-a)$ and $d(f(q), f(p')) = (t+\Delta t)\,\nu(-a)$. This gives $d(f(p), f(p')) = \Delta t\,\nu(-a)$, as claimed. The rest of the proof follows as in Proposition 2.1. □

3 Flat Planes

We now turn to Theorem 1.1. Recall that a metric space X is *δ-hyperbolic*, for some constant $\delta \geq 0$, if for every quadruple $(w,x,y,z) \in X^4$,

$$d(w,y) + d(x,z) \leq \max\{d(w,x) + d(y,z), d(w,z) + d(x,y)\} + 2\delta.$$

If such a δ exists, X is said to be *hyperbolic*. As is well known, for a geodesic metric space this is equivalent to saying that geodesic triangles are slim, in an appropriate sense. It also suffices to consider triangles whose sides are given by a fixed bicombing:

Lemma 3.1. *Let X be a metric space with a map that selects for every pair of points $x, y \in X$ a geodesic segment $[x,y] = [y,x] \subset X$ connecting them. If for every triple $(x,y,z) \in X^3$ the segment $[x,z]$ is contained in the closed $\frac{\delta}{2}$-neighborhood of $[x,y] \cup [y,z]$, then X is δ-hyperbolic.*

Proof. Let $(w, x, y, z) \in X^4$. The union of the closed $\frac{\delta}{2}$-neighborhoods of $[x, y]$ and $[y, z]$ contains $[x, z]$. Likewise, the closed $\frac{\delta}{2}$-neighborhoods of $[z, w]$ and $[w, x]$ cover $[x, z]$. It follows that there is either a pair of points $x' \in [x, y]$ and $z' \in [z, w]$ with $d(x', z') \leq \delta$ or a pair of points $y' \in [y, z]$ and $w' \in [w, x]$ with $d(y', w') \leq \delta$. In the first case,

$$\begin{aligned} d(w, y) + d(x, z) &\leq \big(d(w, z') + \delta + d(x', y)\big) + \big(d(x, x') + \delta + d(z', z)\big) \\ &= d(w, z) + d(x, y) + 2\delta. \end{aligned}$$

Similarly, in the second case, $d(w, y) + d(x, z) \leq d(w, x) + d(y, z) + 2\delta$. □

In particular, a non-hyperbolic X with a bicombing σ contains a sequence of fatter and fatter σ-triangles. The following argument then uses a ruled surface construction together with the cocompact isometric action to produce a collection of mutually asymptotic rays as in Proposition 2.4. This differs from the strategy in [4] and is inspired by the proof for CAT(0) spaces in [6, 7], although we make no use of angles.

Proof of Theorem 1.1. Let X be a proper, cocompact metric space with a consistent bicombing σ. If X contains an isometrically embedded normed plane, then clearly X cannot be hyperbolic.

Suppose now that X is not hyperbolic. We show that then X must contain an embedded normed plane. We continue to write $[x, y]$ in place of $\mathrm{im}(\sigma_{xy})$. By Lemma 3.1 there are sequences of points $y_n^1, y_n^2, y_n^3 \in X$ and $p_n \in [y_n^1, y_n^3]$ such that

$$B(p_n, n) \cap ([y_n^1, y_n^2] \cup [y_n^2, y_n^3]) = \emptyset \tag{3.1}$$

for all integers $n \geq 1$, where $B(p_n, n)$ denotes the closed ball at p_n of radius n. Put $r_n(\cdot) := d(p_n, \cdot)$. For $i = 1, 2$, let x_n^i be a point in $[y_n^i, y_n^{i+1}]$ with minimal distance to p_n, and let $\xi_n^i \colon [0, r_n(x_n^i)] \to X$ be a unit speed parametrization of the segment $[p_n, x_n^i]$ from p_n to x_n^i. Then, for every pair $(i, j) \in \{(1, 1), (1, 2), (2, 2), (2, 3)\}$, we define the "ruled surface"

$$\Delta_n^{i,j} \colon [0, r_n(x_n^i)] \times [0, r_n(y_n^j)] \to X$$

so that $\Delta_n^{i,j}(\cdot, 0) = \xi_n^i$ and, for each $s \in [0, r_n(x_n^i)]$, $\Delta_n^{i,j}(s, \cdot)$ is a constant speed parametrization of the segment $[\xi_n^i(s), y_n^j]$ from $\xi_n^i(s)$ to y_n^j. Thus

$$d(\Delta_n^{i,j}(s, t), \Delta_n^{i,j}(s, t')) = \frac{d(\xi_n^i(s), y_n^j)}{r_n(y_n^j)} |t - t'| \leq 2|t - t'|,$$

because $d(\xi_n^i(s), y_n^j) \leq r_n(x_n^i) + r_n(y_n^j) \leq 2\, r_n(y_n^j)$ by the choice of x_n^i. Note also that by convexity,

$$d(\Delta_n^{i,j}(s, t), \Delta_n^{i,j}(s', t)) \leq d(\xi_n^i(s), \xi_n^i(s')) = |s - s'|. \tag{3.2}$$

It follows that each $\Delta_n^{i,j}$ is 2-Lipschitz, where here and below we equip $\mathbb{R}^2$ with the l_1-metric. Furthermore, putting $s' := r_n(x_n^i)$, we notice that for $0 \le r \le s \le s'$ and $0 \le t \le r_n(y_n^j)$,

$$\begin{aligned} d(\xi_n^i(r), \Delta_n^{i,j}(s,t)) &\ge r_n(\Delta_n^{i,j}(s',t)) - r - d(\Delta_n^{i,j}(s,t), \Delta_n^{i,j}(s',t)) \\ &\ge r_n(x_n^i) - r - (s' - s) \\ &= s - r \end{aligned} \tag{3.3}$$

by the triangle inequality, the choice of x_n^i, and (3.2).

Now we choose a sequence of isometries γ_n of X so that $\gamma_n(p_n) \in K$ for all n and for some fixed compact set K. By (3.1), $r_n(x_n^i), r_n(y_n^j) > n$. Since X is proper, we can extract a sequence $n(k)$ so that each of the four sequences $\gamma_{n(k)} \circ \Delta_{n(k)}^{i,j}$ converges uniformly on compact sets, as $k \to \infty$, to a 2-Lipschitz map

$$f^{i,j} \colon \mathbb{R}_+ \times \mathbb{R}_+ \to X$$

with boundary rays $\xi^i := f^{i,j}(\cdot, 0)$ and $\eta^j := f^{i,j}(0, \cdot)$. Furthermore, for every $s \in \mathbb{R}_+$, $\eta_s^{i,j} := f^{i,j}(s, \cdot)$ is a ray asymptotic to η^j, so $f^{i,j}$ is in fact 1-Lipschitz. Clearly $\{\xi^i\} \cup \{\eta_s^{i,j} : s \in \mathbb{R}_+\}$ is a coherent collection of rays. From the construction we also have that $d(\eta^1(t), \eta^3(t)) = 2t$ for all $t \ge 0$, in particular η^1, η^3 are non-asymptotic. Hence, there is at least one pair (i, j) such that ξ^i, η^j are non-asymptotic. We put $f := f^{i,j}$, $\xi := \xi^i$, $\eta := \eta^j$, and $\eta_s := \eta_s^{i,j}$ for some such pair. We claim that for all $a \in \mathbb{R}$ and $b > 0$, the limit

$$L(a, b) := \lim_{s \to \infty} d(\xi(s + a), \eta_s(b))$$

exists and is strictly positive. Clearly $L(0, b) = b$. If $a > 0$, then $L(a, b) = d_\infty((a, \xi), (b, \eta)) > 0$ by the first part of Proposition 2.4 and since ξ, η are non-asymptotic. If $a < 0$, the same result still shows that $s \mapsto d(\xi(s + a), \eta_s(b))$ is non-increasing on $[|a|, \infty)$, so the limit exists, and $L(a, b) \ge |a|$ as a consequence of (3.3).

Next, for every integer $l \ge 1$, we define the 1-Lipschitz map

$$f_l \colon [-l, \infty) \times \mathbb{R}_+ \to X, \quad f_l(s, t) := f(l + s, t) = \eta_{l+s}(t).$$

Then we choose isometries $\bar\gamma_l$ of X so that $(\bar\gamma_l \circ f_l)(0, 0) \in K$ for all l and for some fixed compact set K. As above, there exists a subsequence $l(k)$ such that the sequence $\bar\gamma_{l(k)} \circ f_{l(k)}$ converges uniformly on compact sets to a 1-Lipschitz map

$$\bar f \colon \mathbb{R} \times \mathbb{R}_+ \to X$$

with boundary line $\bar\xi := \bar f(\cdot, 0)$ and mutually asymptotic rays $\bar\eta_s := \bar f(s, \cdot)$ for $s \in \mathbb{R}$. Again, $\{\bar\xi\} \cup \{\bar\eta_s : s \in \mathbb{R}\}$ is a coherent collection of geodesics. For every $a \in \mathbb{R}$ and $b > 0$, we now have that

$$d(\bar\xi(s + a), \bar\eta_s(b)) = L(a, b) > 0$$

for all $s \in \mathbb{R}$. Hence, by Proposition 2.4, $\bar{f}$ is an isometric embedding with respect to some norm on $\mathbb{R}^2$. Using once more that X is cocompact, we then conclude that X contains an isometrically embedded normed plane. $\square$

It is clear that if X is a $\mathrm{CAT}(0)$ or a Busemann space, then this property is inherited by any isometrically embedded normed plane, thus the corresponding norm must be Euclidean or strictly convex, respectively. We briefly discuss another variant of Theorem 1.1, which happens to have a very short proof, without reference to bicombings. Recall that a metric space X is *injective* (as an object in the metric category with 1-Lipschitz maps as morphisms), if for every metric space B and every 1-Lipschitz map $f\colon A \to X$ defined on a set $A \subset B$ there is 1-Lipschitz extension $\bar{f}\colon B \to X$. By a remarkable result of Isbell [21], every metric space Y has an injective hull $\mathrm{E}(Y)$, thus every isometric embedding of Y into an injective metric space X factors as $Y \subset \mathrm{E}(Y) \to X$ (see Sects. 2 and 3 in [24] for a survey).

Let now $Q = \{w, x, y, z\}$ be any metric space of cardinality four, and suppose that $c := d(w, y) + d(x, z)$ is not less than the maximum of $a := d(w, x) + d(y, z)$ and $b := d(w, z) + d(x, y)$. The injective hull (or the *tight span* [12]) of Q is isometric to the (possibly degenerate) rectangle $[0, \frac{1}{2}(c-a)] \times [0, \frac{1}{2}(c-b)]$ in $(\mathbb{R}^2, \|\cdot\|_1)$ with four segments of appropriate lengths attached at the corners, where the terminal points of these segments correspond to Q. (See Fig. A1 on p. 336 in [12]. It is also worth pointing out that the 1-skeleton of the tight span of Q, viewed as polyhedral complex, is the unique optimal network realizing the metric of Q; see p. 325 in the same paper.) Now the δ-hyperbolicity of Q means precisely that the width (the minimum of the two side lengths) of this l_1-rectangle is not bigger than δ. This has the following easy consequence.

Theorem 3.2. *A proper, cocompact injective metric space X is hyperbolic if and only if X does not contain an isometric copy of $(\mathbb{R}^2, \|\cdot\|_1)$ or, equivalently, of $(\mathbb{R}^2, \|\cdot\|_\infty)$.*

Proof. Suppose that X is not hyperbolic. Then, by the above observation, for arbitrarily large $\delta > 0$ there exists a quadruple $Q \subset X$ whose injective hull contains an isometric copy of $[0, \delta] \times [0, \delta] \subset (\mathbb{R}^2, \|\cdot\|_1)$. Since X is injective, this l_1-square embeds isometrically into X by the respective property of the injective hull. From a sequence of such squares with side lengths tending to infinity we obtain an isometric embedding of the entire l_1-plane, using the fact that X is proper and cocompact. $\square$

4 Semi-simple isometries

In preparation for the proof of Theorem 1.2 we now discuss semi-simple isometries of a metric space X with a (not necessarily consistent) bicombing σ. The main purpose is to establish basic properties regarding sets of

minimal displacement, analogous to those in the case of CAT(0) spaces. Whereas in the latter case a key role is played by the projection onto convex subspaces, we use a barycenter map for finite subsets of X instead, which we first describe. The same tool will be employed again in Sect. 6.

In [15], Es-Sahib and Heinich introduced an elegant barycenter construction for Busemann spaces, which was reviewed and partly improved in a recent paper by Navas [26]. The construction and proofs translate almost verbatim to spaces with bicombings. For finite subsets, the result is as follows.

Theorem 4.1. *Let X be a complete metric space with a bicombing σ. For every integer $n \geq 1$ there exists a map $\mathrm{bar}_n \colon X^n \to X$ such that*

(1) $\mathrm{bar}_n(x_1, \dots, x_n)$ *lies in the closed σ-convex hull of $\{x_1, \dots, x_n\}$;*

(2) $d(\mathrm{bar}_n(x_1, \dots, x_n), \mathrm{bar}_n(y_1, \dots, y_n)) \leq \min_{\pi \in S_n} \frac{1}{n} \sum_{i=1}^n d(x_i, y_{\pi(i)})$*;*

(3) $\gamma\, \mathrm{bar}_n(x_1, \dots, x_n) = \mathrm{bar}_n(\gamma x_1, \dots, \gamma x_n)$ *whenever γ is an isometry of X and σ is γ-equivariant.*

We shall sometimes suppress the subscript n. The construction is such that $\mathrm{bar}_1(x) := x$, $\mathrm{bar}_2(x, y) := \sigma_{xy}(\frac{1}{2}) = \sigma_{yx}(\frac{1}{2})$, and, for $n \geq 3$,

$$\mathrm{bar}_n(x_1, \dots, x_n) = \mathrm{bar}_n(\mathrm{bar}_{n-1}(\mathbf{x}^1), \dots, \mathrm{bar}_{n-1}(\mathbf{x}^n)),$$

where $\mathbf{x}^i := (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$. The proof of Theorem 4.1 is then not very difficult. The more profound observation of [15, 26] is that the above construction can be modified so as to yield a barycenter map on the space of probability measures with finite first moment that is 1-Lipschitz with respect to the 1-Wasserstein metric. We will not use this more elaborate construction in the present paper.

Now we turn to the discussion of isometries. We begin by recalling some standard terminology and basic facts. First, let X be an arbitrary metric space. For any map $\gamma \colon X \to X$ we denote by $d_\gamma(x) := d(x, \gamma(x))$ the *displacement* at a point $x \in X$, and we put

$$|\gamma| := \inf_{x \in X} d_\gamma(x) \quad \text{and} \quad \mathrm{Min}(\gamma) := \{x \in X : d_\gamma(x) = |\gamma|\}.$$

An isometry γ of X is called *parabolic* if $\mathrm{Min}(\gamma)$ is empty and *semi-simple* otherwise. In the latter case, γ is *elliptic* if $|\gamma| = 0$ (that is, γ has a fixed point) and *hyperbolic* if $|\gamma| > 0$.

For an isometry γ, a line $\xi \colon \mathbb{R} \to X$ will be called an *axis* of γ if there exists a $t > 0$ such that

$$\gamma(\xi(s)) = \xi(s + t) \quad \text{for all } s \in \mathbb{R}. \tag{4.1}$$

Then, for $x := \xi(0)$ and any $y \in X$, the triangle inequality gives

$$\begin{aligned} d(x, \gamma^n(x)) &\leq d(x, y) + n\, d_\gamma(y) + d(\gamma^n(y), \gamma^n(x)) \\ &= n\, d_\gamma(y) + 2\, d(x, y) \end{aligned} \tag{4.2}$$

for all $n \geq 1$, where $d(x, \gamma^n(x)) = nt$, thus $t \leq d_\gamma(y)$ and so $d_\gamma(x) = t = |\gamma|$. Hence every isometry γ with an axis is hyperbolic, and all axes of γ are contained in $\mathrm{Min}(\gamma)$.

For the converse, let γ be a hyperbolic isometry of X with $|\gamma| =: t$, let $x \in \mathrm{Min}(\gamma)$, and suppose there is a geodesic $\tau\colon [0, t] \to X$ from x to $\gamma(x)$. Then the curve $\xi\colon \mathbb{R} \to X$ satisfying

$$\xi(nt + s) = \gamma^n(\tau(s)) \quad \text{for all } n \in \mathbb{Z} \text{ and } s \in [0, t] \tag{4.3}$$

is a local geodesic (in fact it preserves all distances less than or equal to t), because ξ is parametrized by arc length and $d(\xi(nt + s), \xi(nt + t + s)) = d_\gamma(\tau(s)) \geq t$. This curve ξ also satisfies (4.1), hence it is an axis of γ if it happens to be a line. This is the case, for example, if X is a Busemann space, as then every local geodesic in X is a geodesic. (If $\eta\colon [a, b] \to X$ is the geodesic from $\xi(a)$ to $\xi(b)$, then the nonnegative function $s \mapsto d(\xi(s), \eta(s))$ is locally convex, hence convex on $[a, b]$, hence identically zero as it vanishes at the endpoints.) Thus every hyperbolic isometry of a Busemann space is axial (compare Chap. 11 in [27]).

The following result shows in particular that this last fact remains true in the more general context of this paper. Recall that a bicombing σ of X is *γ-equivariant*, for an isometry γ of X, if $\gamma \circ \sigma_{xy} = \sigma_{\gamma(x)\gamma(y)}$ for all $(x, y) \in X^2$.

Proposition 4.2. *Let $\gamma\colon X \to X$ be an isometry of a complete metric space X with a γ-equivariant bicombing σ. Then:*

(1) *For all $x, y \in X$ and $n \geq 1$, $|\gamma| \leq \frac{1}{n}\, d(x, \gamma^n x) \leq d_\gamma(y) + \frac{2}{n}\, d(x, y)$.*

(2) *For all $x \in X$, $\lim_{n\to\infty} \frac{1}{n}\, d(x, \gamma^n x) = |\gamma|$.*

(3) *If γ is hyperbolic, then for every $x \in \mathrm{Min}(\gamma)$ there exists an axis of γ through x.*

(4) *If $C \subset X$ is non-empty, σ-convex, and γ-invariant, then $|\gamma| = \big|\gamma|_C\big|$.*

Proof. The second inequality in (1) is just (4.2). For the first, we employ the barycenter construction stated above. Given $x \in X$ and $n \geq 1$, put $\mathbf{x} := (x, \gamma x, \dots, \gamma^{n-1} x)$ and $\gamma\mathbf{x} := (\gamma x, \dots, \gamma^n x)$. Then

$$|\gamma| \leq d(\mathrm{bar}_n \mathbf{x}, \gamma\, \mathrm{bar}_n \mathbf{x}) = d(\mathrm{bar}_n \mathbf{x}, \mathrm{bar}_n \gamma\mathbf{x}) \leq \frac{1}{n}\, d(x, \gamma^n x),$$

where the last two steps use parts (3) and (2) of Theorem 4.1, respectively. The limit formula (2) is an immediate consequence of (1). As for (3), let γ be a hyperbolic isometry with $|\gamma| =: t$, and let $x \in \mathrm{Min}(\gamma)$. Then, for $y = x$, (1) shows that $d(x, \gamma^n x) = nt$ for all $n \geq 1$, thus any curve ξ as in (4.3) is a line. Finally, given any set C as in (4), its closure $\overline{C}$ is still σ-convex and γ-invariant, and furthermore complete. Now fix any $x \in C$ and apply (2) for both γ and $\gamma|_{\overline{C}}$ to conclude that $|\gamma| = \big|\gamma|_{\overline{C}}\big|$. Clearly $\big|\gamma|_{\overline{C}}\big| = \big|\gamma|_C\big|$. $\square$

The following standard result will be used several times in the sequel.

Lemma 4.3. *Let X be a proper metric space, let Γ be a group acting properly and cocompactly by isometries on X, and let $\alpha_1, \dots, \alpha_n \in \Gamma$. Given a sequence of points in X along which the displacement functions $d_{\alpha_1}, \dots, d_{\alpha_n}$ are bounded, there exist a subsequence x_k and isometries $\gamma_k \in \Gamma$ such that $\gamma_k x_k$ converges to a point $z \in X$ and, for every element α of the subgroup $\langle \alpha_1, \dots, \alpha_n \rangle$, $\gamma_k \alpha \gamma_k^{-1} \in \Gamma$ is independent of k and $\lim_{k\to\infty} d_\alpha(x_k) = d_\alpha(y)$ for all points y in the sequence $\gamma_k^{-1} z$.*

In particular, for any $\alpha \in \Gamma$, starting from a minimizing sequence for d_α one gets that $|\alpha| = \lim_{k\to\infty} d_\alpha(x_k) = d_\alpha(y)$ for some point y. This shows that Γ acts by semi-simple isometries (compare Proposition II.6.10 in [7]).

Proof. Since the action is cocompact we may assume, by passing to a subsequence x_k, that there exist $\gamma_k \in \Gamma$ such that $\gamma_k x_k$ converges to a point $z \in X$. By assumption the sequence $d(\gamma_k x_k, \gamma_k \alpha_1 \gamma_k^{-1}(\gamma_k x_k)) = d_{\alpha_1}(x_k)$ is bounded, so $d(z, \gamma_k \alpha_1 \gamma_k^{-1} z)$ is bounded as well. Hence, because the action of Γ is proper, we may pass to a further subsequence in order to arrange that $\gamma_k \alpha_1 \gamma_k^{-1}$ is equal to the same element $\overline{\alpha}_1 \in \Gamma$ for all k. Repeating the argument for $\alpha_2, \dots, \alpha_n$, we arrive at a map $\alpha_i \mapsto \overline{\alpha}_i$ which extends to a homomorphism $\alpha \to \overline{\alpha}$ from $\langle \alpha_1, \dots, \alpha_n \rangle$ into Γ such that $\gamma_k \alpha \gamma_k^{-1} = \overline{\alpha}$ for all k. Now it follows that

$$\lim_{k\to\infty} d_\alpha(x_k) = \lim_{k\to\infty} d_{\overline{\alpha}}(\gamma_k x_k) = d_{\overline{\alpha}}(z) = d_\alpha(y)$$

whenever $y = \gamma_k^{-1} z$ for some k. $\square$

From the above results we obtain a crucial fact for the proof of Theorem 1.2.

Proposition 4.4. *Let X be a proper metric space with a bicombing σ. Let Γ be a group acting properly and cocompactly by isometries on X, and suppose that σ is Γ-equivariant. Then for every finitely generated abelian subgroup A of Γ the set*

$$\mathrm{Min}(A) := \bigcap_{\alpha \in A} \mathrm{Min}(\alpha)$$

is non-empty (and σ-convex, α-invariant for every $\alpha \in A$, and closed).

Proof. For an individual $\alpha \in A$ the set $\mathrm{Min}(\alpha)$ is non-empty, as noted after Lemma 4.3. Now suppose that $B \subset A$ is a finite set such that $\mathrm{Min}(B) := \bigcap_{\beta \in B} \mathrm{Min}(\beta) \neq \emptyset$, and let $\alpha \in A \setminus B$. Note that $\mathrm{Min}(B)$ is σ-convex and furthermore α-invariant, as α commutes with every element of B. Using Proposition 4.2(4) we find a sequence x_k such that $d_\alpha(x_k) \to \big|\alpha|_{\mathrm{Min}(B)}\big| = |\alpha|$ and $d_\beta(x_k) = |\beta|$ for all $\beta \in B$. Applying Lemma 4.3 for the set $B \cup \{\alpha\} \subset \Gamma$ we get a point $y \in \mathrm{Min}(B \cup \{\alpha\})$. This shows that $\mathrm{Min}(B) \neq \emptyset$ for every finite set $B \subset A$. Exhausting A by an increasing sequence of finite subsets we obtain a sequence x_k in X such that for every $\alpha \in A$, the sequence $d_\alpha(x_k)$ is eventually constant with value $|\alpha|$. Applying Lemma 4.3 again, for generators $\alpha_1, \dots, \alpha_n$ of A, we conclude that $\mathrm{Min}(A)$ is non-empty. □

5 σ-Axes

In Proposition 4.2 we showed that every hyperbolic isometry γ of a complete metric space X with a γ-equivariant bicombing σ is axial. It is natural to ask whether γ also admits an axis that is at the same time a σ-line. Such an axis will be called a *σ-axis*. It turns out that the answer to this question is negative in general, see Example 5.4. However, we shall prove in Proposition 5.5 that any group Γ satisfying the assumptions of Theorem 1.2 acts by "σ-semi-simple" isometries, that is, every element has either a fixed point or a σ-axis.

We start with an auxiliary result which will be useful in the proof of Proposition 5.3. In [18], Goebel and Koter proved a fixed point theorem for "rotative" nonexpansive mappings in closed convex subsets of Banach spaces. The argument can easily be adapted to the present context.

Theorem 5.1. *Let Y be a complete metric space with a bicombing σ. Then every 1-Lipschitz map $\varphi\colon Y \to Y$ for which there exist an $n \geq 2$ and $0 \leq a < n$ such that*

$$d(y, \varphi^n(y)) \leq a\, d(y, \varphi(y)) \quad \textit{for all } y \in Y$$

has a fixed point. Furthermore, the fixed point set of φ is a 1-Lipschitz retract of Y.

Proof. Let $\lambda \in (0,1)$ (an appropriate value depending only on a, n will be determined at the end of the proof). For every $x \in Y$, the map sending $y \in Y$ to $[x, \varphi(y)](\lambda)$ is λ-Lipschitz and thus has a unique fixed point $f_\lambda(x) \in Y$ by Banach's contraction mapping theorem. This yields a map $f_\lambda\colon Y \to Y$ with the property that $f_\lambda(x) = [x, \varphi(f_\lambda(x))](\lambda)$ for all $x \in Y$. By convexity,

$$\begin{aligned} d(f_\lambda(x), f_\lambda(y)) &\leq (1-\lambda)\, d(x,y) + \lambda\, d(\varphi(f_\lambda(x)), \varphi(f_\lambda(y))) \\ &\leq (1-\lambda)\, d(x,y) + \lambda\, d(f_\lambda(x), f_\lambda(y)), \end{aligned}$$

so f_λ is 1-Lipschitz. Furthermore, f_λ has the same fixed points as φ, because $[x,\varphi(x)](\lambda) = x$ if and only if $\varphi(x) = x$. By the assumption on φ,

$$\begin{aligned} d(y,\varphi(f_\lambda(y))) &\le d(y,\varphi^n(y)) + d(\varphi^n(y),\varphi(f_\lambda(y))) \\ &\le a\,d(y,\varphi(y)) + d(\varphi^{n-1}(y),f_\lambda(y)). \end{aligned} \tag{5.1}$$

To estimate the last term, note that, again by convexity,

$$\begin{aligned} d(\varphi^m(y),f_\lambda(y)) &\le (1-\lambda)\,d(\varphi^m(y),y) + \lambda\,d(\varphi^m(y),\varphi(f_\lambda(y))) \\ &\le (1-\lambda)m\,d(\varphi(y),y) + \lambda\,d(\varphi^{m-1}(y),f_\lambda(y)), \end{aligned}$$

for $m \in \{1,\dots,n-1\}$. It follows that

$$d(\varphi^{n-1}(y),f_\lambda(y)) \le (1-\lambda)c_\lambda(n)\,d(\varphi(y),y) + \lambda^{n-1}d(y,f_\lambda(y)), \tag{5.2}$$

where $c_\lambda(n) := (n-1) + (n-2)\lambda + \cdots + \lambda^{n-2}$. Combining the fact that $d(y,f_\lambda(y)) = \lambda\,d(y,\varphi(f_\lambda(y)))$ with (5.1) and (5.2) we get

$$d(y,f_\lambda(y)) \le \frac{a + (1-\lambda)c_\lambda(n)}{1-\lambda^n}\,\lambda\,d(y,\varphi(y)).$$

Now let $x \in Y$, and put $y := f_\lambda(x)$. Then $y = [x,\varphi(y)](\lambda)$ and hence $\lambda\,d(y,\varphi(y)) = (1-\lambda)\,d(x,y)$, thus we obtain

$$d(f_\lambda(x),f_\lambda^2(x)) \le \frac{a + (1-\lambda)c_\lambda(n)}{1+\lambda+\cdots+\lambda^{n-1}}\,d(x,f_\lambda(x)).$$

Since the factor on the right converges to $\frac{a}{n} < 1$ for $\lambda \to 1$, there is a $\lambda \in (0,1)$ making it strictly less than 1. Then, for every $x \in Y$, the sequence $k \mapsto f_\lambda^k(x)$ is Cauchy. Since f_λ is 1-Lipschitz, it follows that the limit point $\varrho(x)$ of this sequence is a fixed point of f_λ, hence a fixed point of φ, and ϱ is a 1-Lipschitz retraction of Y onto the fixed point set of φ. □

Now let γ be any isometry of a metric space X with a bicombing σ. We associate with γ the map

$$\varphi_\gamma \colon X \to X, \quad \varphi_\gamma(x) = [\gamma x, \gamma^{-1}x]\big(\tfrac{1}{2}\big).$$

Note that $d(\varphi_\gamma(x),\varphi_\gamma(y)) \le \frac{1}{2}\,d(\gamma x,\gamma y) + \frac{1}{2}\,d(\gamma^{-1}x,\gamma^{-1}y) = d(x,y)$, thus φ_γ is 1-Lipschitz. Our interest in this map comes from the following simple fact.

Lemma 5.2. *Let γ be an isometry of a metric space X with a γ-equivariant consistent bicombing σ, and let $x \in X$ be such that $\gamma(x) \neq x$. Then there exists a σ-axis of γ through x if and only if x is a fixed point of the associated map $\varphi = \varphi_\gamma$.*

Proof. If $\xi\colon X \to \mathbb{R}$ is a σ-axis of γ through x, then clearly $\varphi(x) = x$. Conversely, suppose that x is a fixed point of φ. Put $t := d_\gamma(x)$. Let $\tau\colon [0,t] \to X$ be defined by $\tau(s) = [x, \gamma x](\frac{s}{t})$, and consider the corresponding unit speed curve $\xi\colon \mathbb{R} \to X$ satisfying (4.3). Since $\varphi(x) = x$ and σ is consistent, it follows that ξ is a "local σ-line", in fact every subsegment of length t is σ-convex. Then, as in the case of Busemann spaces, it follows that ξ is a (global) σ-line and hence a σ-axis of x. $\square$

We now show that the translation length $|\varphi_\gamma| = \inf_{x\in X} d_{\varphi_\gamma}(x)$ of φ_γ is always zero, provided the bicombing σ is γ-equivariant.

Proposition 5.3. *Let γ be an isometry of a metric space X with a γ-equivariant bicombing σ. Then for all $x \in X$ and $n \geq 1$,*

$$d(x, {\varphi_\gamma}^n(x)) \leq \sqrt{n}\, d_\gamma(x),$$

and $|\varphi_\gamma| = 0$.

Proof. We assume without loss of generality that X is complete. We write $\varphi := \varphi_\gamma$. Let $x \in X$. For all $m \in \mathbb{Z}$ and $0 \leq n \in \mathbb{Z}$, put $x_{n,m} := \varphi^n(\gamma^m x)$ and $d_{n,m} := d(x, x_{n,m})$. Note that φ and γ commute because σ is γ-equivariant. In particular, for $n \geq 1$, we have $x_{n,m} = [x_{n-1,m-1}, x_{n-1,m+1}](\frac{1}{2})$ and hence

$$d_{n,m} \leq \frac{1}{2}(d_{n-1,m-1} + d_{n-1,m+1}).$$

By induction on n this yields

$$d_{n,m} \leq 2^{-n} \sum_{i=0}^{n} \binom{n}{i} d_{0,m-n+2i}\ .$$

Since $d_{0,m} \leq |m|\, d_\gamma(x)$, we obtain

$$d_{n,0} \leq 2^{-n} \sum_{i=0}^{n} \binom{n}{i} d_{0,2i-n} \leq 2\, d_\gamma(x) \cdot 2^{-n} \sum_{i=0}^{n} \binom{n}{i} \left| i - \frac{n}{2} \right|.$$

As $2^{-n}\binom{n}{i}$ is the probability mass function of a binomial distribution with parameters n and $\frac{1}{2}$ (number of trials and probability of success), let Z be a random variable distributed accordingly. Recall that the mean and variance are $\mathrm{E}[Z] = \frac{n}{2}$, $\mathrm{Var}[Z] = \frac{n}{4}$, hence

$$\begin{aligned} \frac{d_{n,0}}{2\, d_\gamma(x)} &\leq \mathrm{E}[|Z - \mathrm{E}[Z]|] = \mathrm{E}\left[\sqrt{(Z - \mathrm{E}[Z])^2}\right] \\ &\leq \sqrt{\mathrm{E}[(Z - \mathrm{E}[Z])^2]} = \sqrt{\mathrm{Var}[Z]} = \frac{\sqrt{n}}{2} \end{aligned}$$

by Jensen's inequality. Thus $d(x, \varphi^n(x)) = d_{n,0} \leq \sqrt{n}\, d_\gamma(x)$.

For any $c > |\gamma|$, $Y := \{x \in X : d_\gamma(x) \le c\}$ is a non-empty, complete and σ-convex set with $\gamma(Y) = Y$ and, consequently, $\varphi(Y) \subset Y$. Now if $|\varphi|$ was positive, then for some sufficiently large n and for some $a < n$ we would have $d(x, \varphi^n(x)) \le c\sqrt{n} \le a|\varphi| \le a\, d(x, \varphi(x))$ for all $x \in Y$, and Theorem 5.1 would provide a fixed point $y = \varphi(y)$, in contradiction to $|\varphi| > 0$. This shows that $|\varphi| = 0$. □

The following example shows that, in general, the infimum $|\varphi_\gamma| = 0$ need not be attained. The isometry γ we construct is axial, but has no σ-axis.

Example 5.4. Let $X := l_\infty(\mathbb{Z})$ be the Banach space of bounded functions $x\colon \mathbb{Z} \to \mathbb{R}$, with the supremum norm, and consider the affine bicombing $(x, y, \lambda) \mapsto (1-\lambda)x + \lambda y$ (there is in fact no other bicombing on X, see Theorem 1 in [17]). Let $\varrho\colon X \to X$ be the shift map satisfying $\varrho(x)(k) = x(k-1)$ for all $x \in X$ and $k \in \mathbb{Z}$, and let $p \in X$ be defined by $p(k) = 1$ for $k \ge 1$ and $p(k) = 0$ otherwise. The isometry $\gamma\colon X \to X$, $\gamma(x) := \varrho(x) + p$, satisfies $\|\gamma^n(0)\|_\infty = n$ for all $n \ge 1$, so $|\gamma| = d_\gamma(0) = 1$ by Proposition 4.2. Thus γ is hyperbolic and hence axial. The associated map $\varphi = \varphi_\gamma\colon X \to X$ is given by

$$\varphi(x) = \frac{1}{2}(\varrho(x) + \varrho^{-1}(x) - z),$$

where $z := \varrho^{-1}(p) - p$ is the indicator function of 0. Now an $x \in X$ with $\varphi(x) = x$ would have to fulfil $x(0) = \frac{1}{2}(x(-1) + x(1) - 1)$ as well as $x(k) = \frac{1}{2}(x(k-1) + x(k+1))$ for all $k \ne 0$, and it is easy to see that no such bounded function $x\colon \mathbb{Z} \to \mathbb{R}$ exists.

In contrast to this example, the following holds.

Proposition 5.5. *Let X be a proper metric space with a consistent bicombing σ. Let Γ be a group acting properly and cocompactly by isometries on X, and suppose that σ is Γ-equivariant. Then every isometry $\alpha \in \Gamma$ has either a fixed point or a σ-axis.*

Proof. Let $\alpha \in \Gamma$. In view of Lemma 5.2 we just need to show that the associated map $\varphi := \varphi_\alpha$ has a fixed point. Let $r > |\alpha|$. Applying Proposition 5.3 to the complete, σ-convex and α-invariant set $X_r := \{x \in X : d_\alpha(x) \le r\}$, we find a sequence of points in X_r along which the displacement function d_φ tends to zero. By Lemma 4.3 there exist a subsequence x_k and isometries $\gamma_k \in \Gamma$ such that $\gamma_k x_k$ converges to a point $z \in X$ and $\gamma_k \alpha \gamma_k^{-1} =: \overline{\alpha} \in \Gamma$ is constant. Put $\overline{\varphi} := \varphi_{\overline{\alpha}}$. Since σ is γ_k-equivariant, we have that for all $y \in X$,

$$\gamma_k \circ [\alpha^{-1}y, \alpha y] = [\overline{\alpha}^{-1}(\gamma_k y), \overline{\alpha}(\gamma_k y)],$$

hence $d_\varphi(y) = d(\gamma_k y, (\gamma_k \circ \varphi)y) = d_{\overline{\varphi}}(\gamma_k y)$. It follows that

$$d_\varphi(\gamma_k^{-1} z) = d_{\overline{\varphi}}(z) = \lim_{k\to\infty} d_{\overline{\varphi}}(\gamma_k x_k) = \lim_{k\to\infty} d_\varphi(x_k) = 0,$$

thus every $\gamma_k^{-1} z$ is a fixed point of φ. □

6 Flat tori

We now prove Theorem 1.2. Thus, in the following, X denotes a proper metric space with a consistent bicombing σ, equivariant with respect to a group Γ that acts properly and cocompactly by isometries on X, and A is a free abelian subgroup of Γ of rank n. We retain the multiplicative notation for $A \subset \Gamma$, but we fix once and for all an isomorphism $\iota\colon (\mathbb{Z}^n, +) \to (A, \cdot)$. For generic points $a, b \in \mathbb{Z}^n$, the corresponding elements of A will be denoted by $\alpha := \iota(a)$, $\beta := \iota(b)$ without further comment. We write $b_1 = (1, 0, \dots, 0,), \dots, b_n = (0, \dots, 0, 1)$ for the canonical generators of $\mathbb{Z}^n$ and put $\beta_i := \iota(b_i)$. With this convention, we can state the assertion of Theorem 1.2 as follows: there exist a norm $\|\cdot\|$ on $\mathbb{R}^n$ and an isometric embedding $f\colon (\mathbb{R}^n, \|\cdot\|) \to X$ such that

$$\alpha f(p) = f(p + a) \quad \text{for all } p \in \mathbb{R}^n \text{ and } a \in \mathbb{Z}^n. \tag{6.1}$$

This implies that $d(f(p), \alpha^n f(p)) = \|na\|$ for all $n \geq 1$, therefore $\|a\|$ must be equal to the translation length $|\alpha|$ by Proposition 4.2(2). We first show that a norm with this latter property indeed exists. Notice that we already know from Proposition 4.4 that $\mathrm{Min}(A)$ is non-empty.

Lemma 6.1. *There is a unique norm $\|\cdot\|$ on $\mathbb{R}^n$ such that $\|a\| = |\alpha|$ for every $a \in \mathbb{Z}^n$. With respect to the metric on $\mathbb{Z}^n$ induced by this norm, the map $a \mapsto \alpha x$ is an isometric embedding of $\mathbb{Z}^n$ into X for every $x \in \mathrm{Min}(A)$.*

Proof. Define $\|a\| := |\alpha|$ for all $a \in \mathbb{Z}^n$. Then, for every $x \in \mathrm{Min}(A)$ and $a, b \in \mathbb{Z}^n$,

$$\|b - a\| = |\alpha^{-1}\beta| = d(x, \alpha^{-1}\beta x) = d(\alpha x, \beta x),$$

and this is non-zero if $\alpha \neq \beta$, for otherwise $\alpha^{-1}\beta$ would have a fixed point and infinite order as A is free, in contradiction to the action being proper. Furthermore, $\|ma\| = |m|\|a\|$ for $m \in \mathbb{Z}$ because $|\alpha^m| = |m||\alpha|$ by Proposition 4.2. It follows that $\|\cdot\|$ extends uniquely to a norm on $\mathbb{Q}^n$ and then also to a norm on $\mathbb{R}^n$. $\square$

In the following, $\mathbb{R}^n$ (and $\mathbb{Z}^n, \mathbb{Q}^n$) are always equipped with the metric induced by this norm $\|\cdot\|$. The next result will constitute the last step of the proof of Theorem 1.2.

Proposition 6.2. *Assume that there exists a sequence of 1-Lipschitz maps $f_k\colon \mathbb{R}^n \to X$ such that for all $p \in \mathbb{R}^n$ and $i \in \{1, \dots, n\}$,*

$$\lim_{k \to \infty} d(\beta_i f_k(p), f_k(p + b_i)) = 0.$$

Then there is an isometric embedding $f\colon \mathbb{R}^n \to X$ satisfying (6.1).

Proof. First note that the displacement of β_i along the sequence $f_k(0)$ is bounded by $d(\beta_i f_k(0), f_k(b_i)) + d(f_k(b_i), f_k(0))$, where the first term goes to zero by assumption and the second is bounded by $\|b_i\|$. By Lemma 4.3 we may assume, after passing to a subsequence, that there are isometries $\gamma_k \in \Gamma$ such that $\gamma_k f_k(0)$ converges to a point in X and $\gamma_k \alpha \gamma_k^{-1} =: \overline{\alpha} \in \Gamma$ is constant for every $\alpha \in A$. By the Arzelà–Ascoli theorem we may further assume that the sequence of 1-Lipschitz maps $\gamma_k \circ f_k$ converges uniformly on compact sets to a 1-Lipschitz map $h \colon \mathbb{R}^n \to X$. Now, for all $p \in \mathbb{R}^n$ and $a \in \mathbb{Z}^n$, we have that

$$\begin{aligned} d(\alpha\gamma_k^{-1}h(p), \gamma_k^{-1}h(p+a)) &= d(\overline{\alpha}h(p), h(p+a)) \\ &= \lim_{k\to\infty} d(\overline{\alpha}\gamma_k f_k(p), \gamma_k f_k(p+a)) \\ &= \lim_{k\to\infty} d(\alpha f_k(p), f_k(p+a)). \end{aligned}$$

By assumption this last term is zero if $a \in \{b_1, \dots, b_n\}$. Thus, for any fixed k, the map $f := \gamma_k^{-1} \circ h$ satisfies $\alpha f(p) = f(p+a)$ for all $p \in \mathbb{R}^n$ and for all generators $a = b_i$, hence for all $a \in \mathbb{Z}^n$. This property then forces the 1-Lipschitz map f to be isometric on $\mathbb{Z}^n$ because

$$\|a\| = |\alpha| \le d(f(p), \alpha f(p)) = d(f(p), f(p+a)) \le \|a\|$$

for all $p, a \in \mathbb{Z}^n$. Furthermore, every line segment in $\mathbb{R}^n$ connecting two points in $\mathbb{Z}^n$ is embedded isometrically. Since the set of all pairs of points which lie on a common such segment is dense in $\mathbb{R}^n \times \mathbb{R}^n$, we conclude that f is in fact an isometric embedding. $\square$

Now we proceed as follows. First we construct a 1-Lipschitz map $g \colon \mathbb{R}^n \to \mathrm{Min}(A)$ that sends every ray $\mathbb{R}_+ a$ with $a \in \mathbb{Z}^n \setminus \{0\}$ isometrically to a (σ-)ray asymptotic to a σ-axis of α. Then we use a discrete averaging process based on barycenters (Theorem 4.1) to find a sequence of maps satisfying the assumptions of Proposition 6.2.

Proof of Theorem 1.2. We fix a point $x \in \mathrm{Min}(A)$ and define a map $g \colon \mathbb{R}^n \to X$ as follows. First, for $a \in \mathbb{Z}^n \setminus \{0\}$ and $\lambda \in [0,1]$, put

$$g(\lambda a) := \lim_{k\to\infty} [x, \alpha^k x]\big(\tfrac{\lambda}{k}\big).$$

The limit exists by Lemma 5.1 in [11] since the orbit $\langle\alpha\rangle x$ stays within finite distance of some σ-axis of α by Proposition 5.5, and the definition is clearly consistent for distinct representations of the same point. For $a, b \in \mathbb{Z}^n \setminus \{0\}$ and $\lambda \in [0,1]$ we have

$$\begin{aligned} d(g(\lambda a), g(\lambda b)) &= \lim_{k\to\infty} d\big([x, \alpha^k x]\big(\tfrac{\lambda}{k}\big), [x, \beta^k x]\big(\tfrac{\lambda}{k}\big)\big) \\ &\le \lim_{k\to\infty} \frac{\lambda}{k}\, d(\alpha^k x, \beta^k x) \\ &= \lambda\|a - b\|, \end{aligned}$$

in particular g is 1-Lipschitz on $\mathbb{Q}^n \setminus \{0\}$. It follows that g extends uniquely to a 1-Lipschitz map $g\colon \mathbb{R}^n \to X$.

Next, for all integers $k \geq 1$, put $I_k := [-k,k]^n \cap \mathbb{Z}^n$. We want to establish the following estimate of sublinear growth:

$$e(k) := \sup_{a \in I_k} d(g(a), \alpha x) = o(k) \quad (k \to \infty). \tag{6.2}$$

Given an $\varepsilon > 0$, there is a finite set $B \subset \mathbb{Z}^n$ and a constant C such that for every $a \in \mathbb{Z}^n$ there exist a point $b \in B$ and a positive integer m such that $\|a - mb\| \leq \varepsilon\|a\| + C$. For each $b \in B$ we pick a point $y_b \in \mathrm{Min}(A)$ on a σ-axis of β. Then

$$d(g(mb), \beta^m y_b) = \lim_{k\to\infty} d\big([x, \beta^{mk}x]\big(\tfrac{1}{k}\big), [y_b, \beta^{mk}y_b]\big(\tfrac{1}{k}\big)\big) \leq d(x, y_b),$$

hence $d(g(mb), \beta^m x) \leq 2\,d(x, y_b)$. So let $D := \max\{2\,d(x,y_b) : b \in B\}$. Now, for every $a \in \mathbb{Z}^n$, if b and m are as above, we have

$$\begin{aligned} d(g(a), \alpha x) &\leq d(g(a), g(mb)) + d(g(mb), \beta^m x) + d(\beta^m x, \alpha x) \\ &\leq 2\|a - mb\| + D \\ &\leq 2(\varepsilon\|a\| + C) + D. \end{aligned}$$

This clearly yields (6.2).

To conclude the proof we now construct maps that meet the requirements of Proposition 6.2. Define $f_k\colon \mathbb{R}^n \to X$ by

$$f_k(p) := \mathrm{bar}(\{\alpha^{-1}g(p+a) : a \in I_k\}).$$

Since g is 1-Lipschitz, it follows from Theorem 4.1(2) that f_k is 1-Lipschitz as well. For the generators b_i we have

$$\begin{aligned} \beta_i f_k(p) &= \mathrm{bar}(\{\alpha^{-1}g(p+b_i+a) : a \in I_k - b_i\}), \\ f_k(p+b_i) &= \mathrm{bar}(\{\alpha^{-1}g(p+b_i+a) : a \in I_k\}), \end{aligned}$$

the first equality being a consequence of Theorem 4.1(3) and a change of variable. In order to estimate $d(\beta_i f_k(p), f_k(p+b_i))$ we need a pairing of points in I_k with points in $I_k - b_i$. We match $a \in I_k \cap (I_k - b_i)$ with itself and $a \in I_k \setminus (I_k - b_i)$ with $\tilde{a} := a - (2k+1)b_i \in (I_k - b_i) \setminus I_k$. For a pair of the latter type we have

$$\begin{aligned} d(\alpha^{-1}g(p+b_i+a), x) &= d(g(p+b_i+a), \alpha x) \\ &\leq d(g(p+b_i+a), g(a)) + d(g(a), \alpha x) \\ &\leq \|p + b_i\| + e(k) \end{aligned}$$

as well as $d(\tilde{\alpha}^{-1}g(p+b_i+\tilde{a}), x) \leq \|p+b_i\| + e(k+1)$, since $I_k - b_i \subset I_{k+1}$. Thus

$$d(\alpha^{-1}g(p+b_i+a), \tilde{\alpha}^{-1}g(p+b_i+\tilde{a})) \leq 2(\|p+b_i\| + e(k+1)).$$

Since there are $(2k+1)^{n-1}$ such pairs $(a,\tilde{a})$ out of $|I_k| = (2k+1)^n$ pairs in total, we conclude that

$$d(\beta_i f_k(p), f_k(p+b_i)) \le \frac{2(\|p+b_i\| + e(k+1))}{2k+1} \to 0 \quad (k \to \infty)$$

by Theorem 4.1(2) and (6.2). $\square$

We conclude this section with an example illustrating Theorem 1.2. Given X, σ, and an isometric embedding $f\colon V \to X$ of some n-dimensional normed space V as in the theorem, f carries the canonical bicombing $\bar{\sigma}$ on V to a consistent bicombing on the image of f. However, the geodesics $f \circ \bar{\sigma}_{pq}$ will in general not agree with $\sigma_{f(p)f(q)}$. In fact, the following example for $n = 2$ shows that, despite of much flexibility in the construction of f, it may happen that $\mathrm{im}(f)$ is never σ-convex. This stands in contrast to the case $n = 1$ treated in Proposition 5.5.

Example 6.3. Let $w\colon \mathbb{R} \to \mathbb{R}$ be the 1-periodic function satisfying $w(t) = |t|$ for $t \in [-\frac{1}{2}, \frac{1}{2}]$. Define piecewise affine functions $g, \bar{g}\colon \mathbb{R}^2 \to \mathbb{R}$ by $g(s,t) := w(t)$ and $\bar{g}(s,t) := \max\{w(s), w(t)\}$, and consider the set

$$X := \{(s,t,u) \in \mathbb{R}^3 : g(s,t) \le u \le \bar{g}(s,t)\},$$

endowed with the metric induced by the maximum norm on $\mathbb{R}^3$. It follows as in Example 2.2 that X admits a unique consistent bicombing σ and that the lines $\xi, \xi'\colon \mathbb{R} \to X$ defined by $\xi(s) := (s,0,0)$ and $\xi'(t) := (\frac{1}{2}, t, \frac{1}{2})$ are two σ-lines whose traces are contained in the graphs of g and $\bar{g}$, respectively. Clearly $\mathbb{Z}^2$ acts properly and cocompactly by isometries on X via $((z,z'),x) \mapsto x + (z,z',0)$, and the bicombing σ is $\mathbb{Z}^2$-equivariant. Theorem 1.2 now implies that there exist a norm $\|\cdot\|$ on $\mathbb{R}^2$ and an isometric embedding $f\colon (\mathbb{R}^2, \|\cdot\|) \to X$ such that $f(p) + (z,z',0) = f(p + (z,z'))$ for all $p \in \mathbb{R}^2$ and $(z,z') \in \mathbb{Z}^2$. To describe the image of f, let $\varrho\colon X \to (\mathbb{R}^2, \|\cdot\|_\infty)$ denote the 1-Lipschitz projection $(s,t,u) \mapsto (s,t)$. Since the third coordinates of any two points in X differ by at most $\frac{1}{2}$, it follows that $\varrho \circ f$ preserves all distances greater than $\frac{1}{2}$, but this forces $\varrho \circ f$ to be an isometry altogether. Hence $\varrho|_{\mathrm{im}(f)}$ is an isometry as well, and this implies in turn that $\mathrm{im}(f)$ is the graph of a 1-Lipschitz function $h\colon (\mathbb{R}^2, \|\cdot\|_\infty) \to \mathbb{R}$ such that $g \le h \le \bar{g}$ and h is $\mathbb{Z}^2$-periodic. Now $g = \bar{g} = 0$ on $\mathbb{Z}^2$ and $g = \bar{g} = \frac{1}{2}$ on $\mathbb{R} \times (\frac{1}{2} + \mathbb{Z})$. It follows in particular that the image of f contains the sets $\xi(\mathbb{Z})$ and $\xi'(\frac{1}{2} + \mathbb{Z})$ but cannot contain both the points $\xi(\frac{1}{2}) = (\frac{1}{2}, 0, 0)$ and $\xi'(0) = (\frac{1}{2}, 0, \frac{1}{2})$. Thus, no matter how f is chosen, the image of f will not be σ-convex.

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